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Some Comments on Quantum Magnetic Monopoles

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Abstract In this paper we intend to present some path-integral studies in the problem of confinement in the presence of fermionic and scalar magnetic monopole fields through:

1. A Wilson Loop Path-Integral Evaluation associated to an effective second-quantized electromagnetic field generated by chiral abelian point-like monopole magnetic field current.
2. A Path-Integral Bosonization analysis of Quarks fields interacting with Kalb-Ramond fields considered as an effective Disorder Field Theory of a Q.C.D. vacuum of heavier monopoles.
3. Improvements on the Wilson Loops evaluations in the well-known ADHM Antonov-Ebert model for Cooper pairs of point-like fermionic magnetic monopoles.

Keywords Magnetic monopoles · Wilson loops · Confinement

1 Introduction

The question of the existence of Magnetic Monopoles has been a fruitful research path on modern theoretical physics since the appearance of the seminal work of P.M. Dirac [1] on the subject [1]. In the modern framework of Non-Abelian Gauge theories, most of the relevant dynamical questions about the physical modeling of particles interactions are transferred to the difficult and more subtle mathematical analysis of special gauge-field configurations (instantons, merons, strings, magnetic monopoles, etc.) which are expected to constitute the non-perturbative vacuum structure of the underlying Bosonic Yang-Mills Gauge theory. Among those special field configurations, the Magnetic Monopole has been considered as one of the basic hypothetical non-perturbative excitation expected to be connected to practically all non-trivial charge confining dynamical effects occurring on non-abelian Gauge

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theories. This fact is due to the hope that Magnetic Monopoles are the best candidates for explain naturally the (electrical) charge confinement [2–6]. However magnetic monopoles by themselves should not be observed in the particle spectrum as a physical excitation. Note that this last constraint on monopole confinement makes the use of the standard Quantum Field techniques to handle magnetic monopoles dynamics a very difficult task [7–10].

In this paper we address to these dynamical questions on Magnetic Monopole theory by path integrals analysis, specially the technique of four-dimensional chiral bosonization path-integral as earlier proposed by this author [11, 12].

This paper is organized as follows. In Sect. 2, we show how to obtain by a direct evaluation, the area behavior for an abelian Wilson Loop phase Factor in the presence of an effective second quantized electromagnetic field generated by an (condensate) second quantized monopole fermion field, as much as envisaged as an dynamical mechanism in the famous Nambu-Mandelstam propose for the existence of a Meissner effect for magnetic monopoles vacuum condensation in Yang-Mills theory in order to explain the quark-gluon confinement. As a new result of our study, we claim, thus, to have produced a well-defined path integral procedure to prove the electric charge confining in the presence of a quantum dynamics of magnetic monopoles, with a Fermi-Dirac statistics.

In Sect. 3, we exactly analyze by path-integrals techniques the quantum field dynamics of (massless) fermions field interacting with Kalb-Ramond tensor fields, expected to represent dynamically quark fields interacting with rank-two tensor field, with the later field representing the disorder field of a vacuum structure formed by condensation of magnetic monopoles [7–9]. We show, thus, that it is ill-defined to associated physical observables LSZ interpolating fields for the fermion fields in the theory as consequence of the explicitly Bosonized structure formulae obtained for the matter excitations interacting with rank-two tensor fields through a spin orbit coupling with the Kalb Ramond field strenght, which by its turn provides another support for electrical charge confining in the presence of magnetic monopoles.

Finally in Sect. 4, we present some improvements (originals results of ours on the Wilson Loops evaluation in the context of the Antonov-Ebert dual path integral associated to the dual Abelian Higgs Model of [9].

2 The Abelian Confinement in Presence of Magnetic Monopoles, a Wilson Loop Gauge Invariant Path-Integral Evaluation

Let us start this section by considering the Euclidean path integral average associated to a $U(1)$-abelian field $A_\mu(x)$ whose dual strength field intensity has a second quantized massless magnetic monopole as a (chiral) electromagnetic source

$$\begin{aligned}\langle W[C_{(R,T)}]\rangle = \int D^F[A_\mu]D^F[\Omega]D^F[\overline{\Omega}]\delta^{(F)}[\partial^*_\mu F^{\mu\nu}(A)) - (g\overline{\Omega}\gamma^\nu\gamma^5\Omega)]\\ \times \exp\left(-\frac{1}{2}\int d^4x(\Omega,\overline{\Omega})\begin{bmatrix}0 & (i\partial\!\!\!/)\\ (i\partial\!\!\!/) & 0\end{bmatrix}\begin{pmatrix}\Omega\\ \overline{\Omega}\end{pmatrix}\right)\\ \times \exp\left(ie\oint_{C_{(R,T)}} A_\mu(x^\alpha)dX_\mu\right). \end{aligned}\tag{1a}$$

Here $(\Omega,\overline{\Omega})(x)$ are the Euclidean Fermion (second-quantized) point-like fundamental monopole fields with g denoting the magnetic charge which by its turn is supposed to be related to the $U(1)$-electric charge e by the Dirac quantization relation $eg = \frac{n}{2}$ (with $n \in \mathbb{Z}$). $W[C] = \exp\{ie \oint_{C_{(R,T)}} A_\mu dX_\mu\}$ is the $U(1)$-Wilson Loop phase factor defined by the (Euclidean) space-time trajectory of two static electric carrier external charges interacting with the fluctuating $A_\mu(x)$ field generated by the (fluctuating) second quantized magnetic monopole fermionic source (see the constraint on (1a)). Note that $C_{(R,T)}$ is the boundary of the square $S_{(R,T)}$ below

$$C_{(R,T)} = \partial S_{(R,T)}; \qquad S_{(R,T)} = \left\{ (x_0, x_1) \in R^2; -\frac{T}{2} \le x_0 \le +\frac{T}{2};\ -\frac{R}{2} \le x_1 \le +\frac{R}{2} \right\} \subset R^4. \tag{1b}$$

It is worth call the reader attention that the above written quantum Wilson Loop associated to static quarks charges can be physically replaced by the complete Generating functional of the second quantized Quark fields interacting with the Monopole Generated Electromagnetic field, namely

$$\begin{aligned} Z[\eta, \overline{\eta}] = \int & D^F[A_\mu] D^F[\Omega] D^F[\overline{\Omega}][\delta(*\partial_\mu F^{\mu\nu}(A) - (g\overline{\Omega}\gamma^\nu\gamma^5\Omega)] \\ & \times \exp\left(-\frac{1}{2} \int d^4x (\Omega, \overline{\Omega}) \begin{bmatrix} 0 & i\not\partial \\ i\not\partial & 0 \end{bmatrix} \begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix} \right) \\ & \times \exp\left(-\frac{1}{2} \int d^4x (\psi, \overline{\psi}) \begin{bmatrix} 0 & i\not\partial + \not A \\ (i\not\partial + \not A)^* & 0 \end{bmatrix} \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix} \right) \\ & \times \exp\left(i \int d^h x (\psi, \overline{\psi}) \begin{pmatrix} \overline{\eta} \\ \eta \end{pmatrix} \right). \end{aligned} \tag{1c}$$

For static charges (1c) reduces to (1a) as it is showed in [13].

In order to evaluate the path-integral (1a) from the physical point of view of an effective field theory [11], we should consider firstly re-write the magnetic monopole axial current constraint in (1) by means of an axial-vectorial Lagrange multiplier field $\lambda_\mu(x)$, namely:

$$\begin{aligned} \langle W[C_{(R,T)}] \rangle = \Bigg\{ \int & D^F[A_\mu] D^F[\Omega] D^F[\overline{\Omega}] D^F[\lambda_\mu] \\ & \times \exp\left(i \int d^4x [\lambda_\nu(\partial^*_\mu F^{\mu\nu}(A) - g\overline{\Omega}\gamma^\nu\gamma^5\Omega)](x) \right] \\ & \times \exp\left[-\frac{1}{2} \int d^4x (\Omega, \overline{\Omega}) \begin{bmatrix} 0 & (i\not\partial) \\ (i\not\partial) & 0 \end{bmatrix} \begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix} \right) \\ & \times \exp\left(ie \int_{C_{(R,T)}} A_\mu dX_\mu \right) \Bigg\} \end{aligned} \tag{2}$$

At this point we follow well known studies in the literature in order to give a correct meaning for the effective field theory associated to magnetic monopoles through non perturbative limit in the monopoles Fermionic determinants [11]. It is an expected standard result in the subject that the non-perturbative limit of magnetic monopole dynamics should generates an auxiliary Gauge field mass term, namely

$$\begin{aligned} & |\det(i\not\partial + g\gamma^5 \not\chi_\mu)|^2_{\text{Low-energy}} \\ & \quad \cong \exp\left\{ -\frac{1}{2}(\Lambda \cdot g^2) \int d^4x (\lambda_\mu(x))^2 \right\}. \end{aligned} \tag{3}$$

Note that the appearance through a non-perturbative vacuum monopole scale Λ of a mass term for the auxiliary vector field $\lambda_\mu(x)$ which by its turn, should signals the expected dynamical breaking of the $U(1)$-axial gauge invariance (with opposite parity [10, 11]) of this (non-physical) vectorial field by the phenomenon of dimensional transmutation on the adimensional g-coupling constant. This result indicates strongly the dynamical breaking of the $U(1)$-axial symmetry of the fermionic magnetic monopole second quantized field $\{\Omega(x), \overline{\Omega}(x)\}$.

After inserting (3) into (2) and by realizing the Gaussian λ_μ-field path integral, we are led to consider the effective fourth-order Wilson Loop path integral average for (1) as the effective limit on the magnetic monopole degrees of freedom integrated out, namely:

$$\begin{aligned}\langle W[C_{(R,T)}]\rangle = \bigg\{\int D^F[A_\mu(x)]\delta^{(F)}(\partial_\mu A_\mu) \\ \times \exp\left(-\frac{1}{2(g^2\Lambda)}\int d^4x(A_\mu[(-\partial^2)^2]A_\mu)(x)\right) \\ \times \exp\left(ie\int_{C_{(R,T)}} A_\mu dX_\mu\right)\bigg\}.\end{aligned} \tag{4}$$

The static inter-quark linear risen potential can be obtained from (4) by using the dimensional regularization scheme of Bollini-Giambiagi for evaluating the Feynman-diagrams integrals as it is exposed in details on refs. [13, 14]. It yields the expected linear raising confining potential in our phenomenological model

$$\begin{aligned}V(R) &= (e^2\cdot g^2)(\Lambda)R \\ &= \overline{A}\frac{n^2}{4}(\Lambda)\cdot R = \overline{A}\left(\frac{n^2}{2\pi\alpha'}\right)R = \alpha_{\text{eff}}(N^2)R.\end{aligned} \tag{5}$$

Here $\overline{A}$ is a model-calculational positive adimensional constant, which details will not be needed for our study, and α' denotes the Regge Slope parameter associated to the non-perturbative vacuum scale $\Lambda = \Lambda_{QCD} \sim \frac{1}{2\pi\alpha'}$. It is worth call the reader attention that we have obtained somewhat the infinite quantized number of parallel Regge trajectories from the Dirac topological quantization rule for electric and magnetic charges as it is suggested in the effective Regge slope parameter $\alpha_{\text{eff}}(n^2) = n^2/2\pi\alpha'$.

Thus we see that the effective path integral (1) for the Wilson Loop in the presence of an electromagnetic field generated by a heavy quantum monopole leads naturally to a dynamics of Wilson Loop area behavior for the electrical charges in the theory, a result obtained by us explicitly through an exactly gauge invariant path-integral evaluation.

3 Monopoles Interacting with Kalb-Ramond Fields through Spin-Orbit Coupling

In the last years, Kalb-Ramond field theory has been widely studied as an alternative dynamical quantum field scheme to the Higgs mechanism, as well as in relation to the dynamics of strings in the problem of string representation for Q.C.D. at large number of colors as a dynamical disorder field representing the effects of existence of magnetic monopoles [2–9]. The basic formalism used to analyze such Kalb-Ramond non-perturbative quantum dynamics has been the path-integral formalism, which has shown itself to be a very powerful procedure to understand correctly the different phases of the associated Kalb-Ramond Quantum Field Theory [15–18].

One important problem in those Path-integral studies, still missing in the literature, is that one related to the presence of interacting dynamical fermions (simulating second quantized matter fields) in the Kalb-Ramond Gauge theory. In this Sect. 3 we shall describe the extension of previous path-integral dualization-bosonization studies [19] to the case of Fermionic matter coupling through a spin-orbit field quantum interaction as it is expected to be relevant to describe the interacting physics of quarks and magnetic monopoles.

Let us start by considering the Abelian Kalb-Ramond first order action but now in the presence of massless dynamical fermions in the four-dimensional Euclidean world.

$$S[H,B,\psi,\overline{\psi}]=\int_{R^4} d^4x\left\{\frac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}-\frac{1}{6}H^{\lambda\mu\nu}\partial_{[\lambda}B_{\mu\nu]}+\overline{\psi}(i\partial\!\!\!/+ig\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}H_{\alpha\beta\mu})\psi\right\}. \tag{6}$$

Here the dynamical fields are the independent three-form H, the KR gauge field B and the Dirac fermion fields $(\psi,\overline{\psi})$.

We shall apply the bosonization procedure in the path-integral framework through the following theory's generating functional (normalized to unity)

$$\begin{aligned}Z[J,\eta,\overline{\eta}]=\int & D^F[H]D^F[B]D^F[\psi]D^F[\overline{\psi}]\\ &\times\exp\{-S[H,B,\psi,\overline{\psi}]\}\\ &\times\exp\left\{-i\int_{R^4}d^4x(\overline{\eta}\psi+\overline{\psi}\eta+J_{\mu\nu}B^{\mu\nu})(x)\right\}.\end{aligned} \tag{7}$$

It is worth call the reader attention that the Path-integral (7) is invariant under the KR gauge symmetry, provide the external source current $J_{\mu\nu}$ is chosen to be divergence free and our proposed action term related to the direct interaction of the quantum fermionic matter with the Kalb-Ramond gauge field through its strenght three-form H—the spin orbit fermion interaction (see (6)).

The Path-Integral Bosonization analysis proceeds as usually by integrating exactly out the Kalb-Ramond gauge potential field which produces as a result the delta functional [19].

$$\begin{aligned}Z[J,\eta,\overline{\eta}]=\int & D^F[H]D^F]\psi]D^F[\overline{\psi}]\delta^{(F)}(\partial_\lambda H^{\lambda\mu\nu}-J^{\mu\nu})\\ &\times\exp\left\{-\int_{R^4}d^4x\left[\frac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}+\overline{\psi}(i\partial\!\!\!/+ig\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}H_{\alpha\beta\mu})\psi\right](x)\right\}.\end{aligned} \tag{8}$$

Let us note that the delta functional integrand inside of the path integral (8) imposes the classical equations of motion on the three-form Kalb-Ramond strenght H which by its turn can be exactly solved by the Rham-Hodge theorem in terms of the effective dual scalar axion (zero-form) dynamical degree of freedom in the KR theory defined in a space-time topologically trivial as considered in our path integral (8)

$$H_{\lambda\mu\nu}=g\varepsilon^{\lambda\mu\nu\rho}\partial_\rho\vartheta+\partial^{[\lambda}\frac{1}{\partial^2}J^{\mu\nu]}. \tag{9}$$

At this point we re-write the effective action (8) in a four-dimensional bosonized chiral action [20, 21]

$$Z[J,\eta,\overline{\eta}]=\int D^F[\vartheta]$$

$$\times \exp\left\{-\frac{1}{2}\int_{R^4} d^4x \left[g^2\partial_\mu\vartheta\partial^\mu\vartheta + \frac{1}{2}J^{\mu\nu}\left(-\frac{1}{\partial^2}\right)J_{\mu\nu}\right](x)\right\}$$
$$\times \int D^F[\psi]D^F[\overline{\psi}]\exp\left\{-\frac{1}{2}\int_{R^4} d^4x(\overline{\psi}e^{ig\gamma_5\vartheta}\ \partial\!\!\!/ e^{ig\gamma_5\vartheta}\psi)(x)\right\}$$
$$\times \exp\left\{-\frac{1}{2}\int_{R^4} d^4x\left(ig\overline{\psi}\left[\gamma^\alpha\gamma^\beta\gamma^\rho\partial^{[\alpha}\frac{1}{\partial^2}J^{\beta\rho]}\right]\right)\psi\right\}(x)$$
$$\times \exp\left\{-i\int_{R^4} d^4x(\psi\overline{\eta}+\overline{\psi}\eta)(x)\right\}. \tag{10}$$

After considering the chiral-fermion field variable change on the fermionic path-integral term of (10)

$$\overline{\psi} = \overline{\chi}e^{-ig\gamma_5\vartheta}, \tag{11a}$$
$$\psi = e^{-ig\gamma_5\vartheta}\chi \tag{11b}$$
$$D[\psi]D[\overline{\psi}] = D[\chi]D[\overline{\chi}]\frac{\det[e^{ig\gamma_5\vartheta}\ \partial\!\!\!/ e^{ig\gamma_5\vartheta}]}{\det[\partial\!\!\!/]}$$
$$= D[\chi]D[\overline{\chi}]J[\vartheta], \tag{11c}$$

we obtain the exactly bosonized path-integral representation for the KR first order theory as given by (7), namely:

$$Z[J,\eta,\overline{\eta}] = \int D^F[\vartheta]D[\chi]D[\overline{\chi}]J[\vartheta]$$
$$\times \exp\left\{-\int_{R^4} d^4x\left[\frac{g^2}{2}\partial_\mu\vartheta\partial^\mu\vartheta - \frac{1}{2}J^{\mu\nu}(\partial^2)^{-1}J_{\mu\nu}\right](x)\right\}$$
$$\times \exp\left\{-\frac{1}{2}\int_{R^4} d^4x(\overline{\chi}\ \partial\!\!\!/\chi)(x)\right\}$$
$$\times \exp\left\{-\frac{1}{2}ig\int_{R^4} d^4x\left(\overline{\chi}\left(\gamma^\alpha\gamma^\mu\gamma^\nu\partial^{[\alpha}\frac{1}{\partial^2}J^{\mu\nu]}\right]\chi\right)(x)\right\}$$
$$\times \exp\left\{-i\int_{R^4} d^4x\left(\overline{\chi}e^{-ig\gamma_5\vartheta}\eta + \overline{\eta}e^{-ig\gamma_5\vartheta}\chi\right)(x)\right\}, \tag{12}$$

here the functional Fermion Jacobian (11c) has been exactly evaluated in refs. [20, 21]:

$$J_\varepsilon[\vartheta] = \exp\left\{\frac{g^2}{4\pi^2\varepsilon}\int_{R^4} d^4x(\partial_\mu\vartheta)^2(x)\right\}$$
$$\times \exp\left\{-\frac{g^2}{4\pi^2}\int_{R^4} d^4x(\partial^2\vartheta)(\partial^2\vartheta)(x)\right\}$$
$$\times \exp\left\{\frac{g^4}{12\pi^2}\int_{R^4} d^4x[\vartheta(\partial_\mu\vartheta)^2(-\partial^2\vartheta)](x)\right\}. \tag{13}$$

As a first remark to be made on the above written result we note that its first term has the effect of formally inducing a renormalisation of the g-charge after the cutt-off removing

$\varepsilon \to 0$ on the complete result (7), namely

$$g^2_{\mathrm{bare}}(\varepsilon)\left(1+\frac{1}{4\pi^2\varepsilon}\right)=g^2_{\mathrm{ren}}. \tag{14}$$

An another important physical result coming from the set (12)–(14) is the explicitly fermionic matter asymptotic freedom as can be see directly from the factorized–decoupled form of the full interacting matter fermionic propagator, namely

$$\frac{1}{(i)^3}\frac{\delta Z[\eta,\overline{\eta},J]}{\delta\eta_\alpha(x)\delta\eta_\beta(y)}\bigg|_{\substack{J=0\\ \eta=\overline{\eta}=0}}=S_{\alpha\beta}(x-y)\times F(x,y) \tag{15}$$

with $S_{\alpha\beta}(x-y)$ denoting the free fermion propagator and the (decoupled) Kalb-Ramond form factor being given exactly by the (perturbative finite) fourth-order ϑ-path integral as remarked above.

$$\begin{aligned}F(x,y)=&\int D^F[\vartheta]e^{-\frac{1}{2}g^2_{\mathrm{ren}}\int_{R^4}(\partial_\mu\vartheta)^2(x)d^4x}\\&\times e^{-\frac{g^2_{\mathrm{ren}}}{4\pi^2}\int_{R^4}(\partial^2_\mu\vartheta)^2(x)d^4x}\\&\times e^{+\frac{g^2_{\mathrm{ren}}}{4\pi^2}\int_{R^4}[\vartheta(\partial_\mu\vartheta)^2(-\partial^2_\mu\vartheta)](x)d^4x}\\&\times\{(\exp -ig_{\mathrm{ren}}\gamma_5\vartheta(x))(\exp -ie_{\mathrm{ren}}\gamma_5\vartheta(y))\}\end{aligned} \tag{16}$$

which goes to 1 in the high energy limit of $|x-y|\to 0$ as a result of the path-integral super renormalizability associated to the effective axion scalar dual Kalb-Ramond theory (6) [the well-known phenomenon of asymptotic freedom in confining gauge theories]. A low energy study of the form-factor (16) has been carried out in refs. [20, 21] (Appendix). There, we have suggested that these bosonized fermionic fields do not possesses LSZ interpolating fields, since the associated two-point Euclidean correlation function (15) defines Wightman functions which are ultra-distributions in Jaffe Distributional Spaces and not in the usual Schwartz Tempered Distributional Spaces naturally associated to the existence of LSZ interpolating fields (a well defined Scattering Matrix) in the quantum field theory (7).

A calculational remark to be made at this point of our paper is related to the straightforward exactly solubility for the Macroscopic radiative corrections evaluations of the Kalb-Ramond gauge potential propagator

$$\begin{aligned}&\frac{1}{i^2}\frac{\delta^2[J,\eta,\overline{\eta}]}{\delta J_{\mu\nu}(x)\delta J_{\alpha\beta}(y)}\bigg|_{\substack{\eta-\overline{\eta}=0\\ J=0}}\\&=\langle B_{\mu\nu}(x)B_{\alpha\beta}(y)\rangle\\&=(-\partial^2)^{-1}(x,y)+e^2_{\mathrm{ren}}\int d^4zd^4z'(-\partial^2)^{-1}(z-x)(-\partial^2)^{-1}(z-y)\\&\quad\times\partial^{[\lambda}_z\partial^{\lambda'}_{z'}\langle(\overline{\chi}(z)(\gamma^\lambda\gamma^{[\mu}\gamma^{\nu]})\chi(z))(\overline{\chi}(z')(\gamma^{\lambda'}\gamma^{[\alpha}\gamma^{\beta]})\chi(z'))\rangle^{(0)},\end{aligned} \tag{17}$$

here $\langle\ \rangle^{(0)}$ denotes the free fermion average path integral

$$\langle\ \rangle^{(0)}=\int D(\chi]D[\overline{\chi}]e^{-\frac{1}{2}\int_{R^4}d^4x(\overline{\chi}\partial\!\!\!/\chi)(x)}. \tag{18}$$

The exactly evaluation of the quantum correction (17) is standard and can be easily obtained by just using the well-known Dirac matrixes relationship

$$\gamma^\lambda \gamma^\mu \gamma^\nu = (S_{\lambda\mu\nu\sigma} + \varepsilon_{\lambda\mu\nu\sigma}\gamma_5)\gamma^\sigma, \tag{19}$$

$$S_{\lambda\mu\nu\sigma} = (\delta_{\lambda\mu}\delta_{\nu\sigma} + \delta_{\mu\nu}\delta_{\lambda\sigma} - \delta_{\lambda\nu}\delta_{\mu\sigma}). \tag{20}$$

The above exposed results concludes our Sect. 3 these path-integral method studies on the four-dimensional exactly path-integral Bosonization of our abelian interacting KR field.

4 Some Comments on the Path-Integral Wilson Loop Evaluation in the Dual Abelian Higgs Model of Antonov-Ebert

In this last section of our study, we intend to present some calculational improvements on the Wilson Loop path integral evaluation in the context of the usual Abelian Higgs Model through the framework of path integral duality transformation as exposed in details on the Antonov & Ebert paper in ref. [9].

Let us briefly describe the path integral duality of the extended Dual Abelian Higgs Model of Antonov-Ebert.

As a first step in such analysis, one starts from the following phenomenological expression for the partition functional of the model

$$Z(\lambda) = \int |\Phi| D^F |\Phi| D^F B_\mu D^F \theta \exp\left\{-\int_{R^4}\left[\frac{1}{4}(F_{\mu\nu} - F^E_{\mu\nu})^2 + \frac{1}{2}|D_\mu\Phi|^2 + \lambda(|\Phi|^2 - \eta)^2\right]\right\}, \tag{21a}$$

where $\Phi(x) = |\Phi(x)| \exp|\theta(x)|$ is an disorder scalar Higgs field of the Magnetic Monopoles "Cooper pairs" $(\Omega\overline{\Omega})(x)$ (see (1a)) and $F^E_{\mu\nu}(x)$ is the dual electromagnetic field generated by the external static "quarks" source (the loop $X_\mu(\sigma)$ on (1a)).

It is thus argued in details on the above cited paper of Antonov & Ebert that the partition functional (21) has the following stringy representation in the London phenomenological $\lambda \to +\infty$ limit (the equivalent of our London large mass M limit—(2)–(3). Namely

$$Z^{\mathrm{eff}}(\infty) \sim \int_{\partial X^\mu(\xi) = C^\mu} D^F X_\mu(\xi) \exp\left\{-\pi^2 \int_\Sigma d\sigma_{\lambda\nu}(x) \int_\Sigma d\sigma_{\mu\rho}(y) D^{\lambda\nu,\mu\rho}(|x-y|)\right\}. \tag{21b}$$

Here $X^\mu(\xi)$ parametrizes the string world-sheet Σ possessing as boundary the quark source loop $C^\mu \equiv X_\mu(\sigma)$ and the Antonov-Ebert propagator of the Kalb-Ramond field duality in this London limit is exactly given in momentum space by

$$D^{\lambda\nu,\mu\rho}(|x-y|) = D^{\lambda\nu,\mu\rho}_{(1)}(|x-y|) + D^{\lambda\nu,\mu\rho}_{(2)}(|x-y|), \tag{21c}$$

where

$$D^{\lambda\nu,\mu\rho}_{(1)}(|x|) = (\delta_{\lambda\mu}\delta_{\nu\rho} - \delta_{\mu\nu}\delta_{\lambda\rho})\frac{C_1}{|x|}, \tag{21d}$$

$$D^{\lambda\nu,\mu\rho}_{(2)}(|x|) = \frac{C_2}{|x|^2}\left\{\left[\frac{K_1(m|x|)}{|x|} + \frac{m}{2}(K_0 + K_1)(m|x|)\right]\right.$$

$$\times (\delta_{\lambda\mu}\delta_{\nu\rho} - \delta_{\mu\nu}\delta_{\lambda\rho})$$
$$+ \frac{1}{2|x|}\left[3\left(\frac{m^2}{4} + \frac{1}{|x|^2}\right)K_1(m|x|) + \frac{3m}{2|x|}(K_0 + K_2)(m|x|) + \frac{m^2}{4}K_3(m|x|)\right]$$
$$\left.\times (\delta_{\lambda\rho}x_\mu x_\nu + \delta_{\mu\nu}x_\lambda x_\rho - \delta_{\mu\lambda}x_\nu x_\rho - \delta_{\nu\rho}x_\mu x_\lambda)\right\} \tag{21e}$$

with K_i denoting the relevant usual Modified Bessel functions, m is the mass of the dual gauge bosons generated by the Higgs mechanism and C_1 and C_2 are model calculational constants.

At this point, we argue that all the above pointed out duality derivation holds true only for small string world-sheet deviations from the minimal surface $C^\mu = \partial X^\mu(\xi)$ since we have Frozen the radial part of the monopole disorder field to its fixed v.e.v η. A very important and direct consequence of this remark of ours is that one can safely substitute the somewhat formal Feynman path measure on the string vector position effective partition functional (21b) by the so-called extrinsic space-times vorticity tensor current defined as

$$\Sigma_{\mu\nu}(x) \equiv \int_\Sigma d\sigma_{\mu\nu}(x)\delta(x - x(\xi))$$
$$\equiv \int_\Sigma d^2\xi\, J^{\mu\nu}(\xi)\delta(x - X(\xi))\sqrt{g(X(\xi))},$$
$$\text{where } J^{\mu\nu}(\xi) = \varepsilon^{ab}(\partial_a X^\mu \partial_b X^\nu)(\xi) \tag{21f}$$

is the (non-normalized to unity) string world-sheet extrinsic orientation area tensor.

The argument for the validity of such path-integral dynamical degree replacement is the following.

For small deviations from the minimal area string world-sheet $X^\mu_{c\ell}(\xi)$, we have the usual functional metric decomposition

$$(X^\mu(\xi) = X^\mu_{c\ell}(\xi) + \varepsilon Y^\mu(\xi) + O(\varepsilon)),$$

$$\int_\Sigma d^2\xi\sqrt{g(x^\mu_{c\ell})}(\delta J^{\mu\nu}(\delta J^{\mu\nu})(\xi)$$
$$\sim \int_\Sigma d^2\xi\sqrt{g(X^\mu_{c\ell})}\big\{\big[\varepsilon^{ab}(\partial_a X^\mu_{c\ell})(\partial_b \delta Y^\nu) + \varepsilon^{ab}(\partial_a \delta Y^\mu)(\partial_b X^\mu_{c\ell})\big]$$
$$\times\big[\varepsilon^{ab}(\partial_a X^\mu_{c\ell})(\partial_b \delta Y^\nu) + \varepsilon^{ab}(\partial_a \delta Y^\mu)(\partial_b X^\mu_{c\ell})\big]\big\} + O(\varepsilon^4) \tag{22}$$

which straightforwardly leads to the following volume element functional change

$$D^F[J^{\mu\nu}(\xi)] \equiv \prod_{(\xi)}(\delta J^{\mu\nu}(\xi)) = \left(\prod_{(\xi)}\delta x^\mu(\xi)\right)\cdot\ \det{}^{\frac{1}{2}}_{(\xi,\mu,\nu)}[\mathcal{L}^{\mu\nu}(x^\mu_{c\ell})], \tag{23}$$

where $\mathcal{L}^{\mu\nu}(X^\mu_{c\ell})$ is a second-order elliptic operator depending only on the classical minimal-area string configuration $X^\mu_{c\ell}(\xi)$, so canceling itself when one is realising path-integral averages with the partition functional (21b).

Now it is straightforward to see that one can replace on basis of the above expected small string world-sheet deviations the average over the string vector position by the string vorticity degree of freedom, namely

$$\int_{R^4} (\delta\Sigma^{\mu\nu}(x)\cdot\delta\Sigma_{\mu\nu}(x))d^4x \sim \overline{C}\int_{\Sigma}(\delta J^{\mu\nu}(\xi)\cdot\delta J_{\mu\nu}(\xi))d^2\xi \tag{24}$$

with $\overline{C}$ denoting an over-all (cut-off dependent) constant which cancels with itself when evaluating path-integral averages [13, 14].

The important consequence of the above analyzed variable change of the string vector position variable by the string extrinsic vorticity field is affording the exactly path-integral solubility of the generating functional of the strenght of the usual gauge field A_μ in the Abelian Higgs Model with the following result

$$\begin{aligned} Z[S_{\alpha\beta}] &= \exp\left(-\int d^4x S^2_{\mu\nu}\right) \\ &\quad\times\int D^F[\Sigma^{\gamma\zeta}(x)]\times\exp\left(-4\pi ie\int d^4x(S_{\mu\nu}\Sigma^{\mu\nu})(x)\right) \\ &\quad\times\exp\left\{-\int d^4xd^4y\left(\pi\Sigma_{\lambda\nu}(x)-\frac{i}{e}S_{\lambda\nu}(x)\right)D^{\lambda\nu,\mu\rho}(|x-y|)\right. \\ &\quad\left.\times\left(\pi\Sigma_{\mu\rho}(y)-\frac{i}{e}S_{\mu\rho}(y)\right)\right\}. \end{aligned} \tag{25}$$

For instance:

$$\begin{aligned} &\left\langle\left(\frac{1}{2}\varepsilon^{\lambda\nu\alpha\beta}F_{\alpha\beta}\right)(x)\left(\frac{1}{2}\varepsilon^{\mu\rho\alpha'\beta'}F_{\alpha'\beta'}\right)(x)\right\rangle \\ &\quad\equiv\frac{1}{Z(0)}\frac{\delta^2 Z[S_{\gamma\lambda}]}{\delta S_{\lambda\nu}(x)\delta S_{\mu\rho}(y)}\bigg|_{S_{\gamma\lambda}\equiv 0} \\ &\quad=(\delta_{\lambda\mu}\delta_{\nu\rho}-\delta_{\lambda\rho}\delta_{\mu\nu})\delta^{(4)}|x-y|+\frac{2}{e^2}D^{\lambda\nu,\mu\rho}(|x-y|) \\ &\quad\quad-4\pi^2\left\langle\left[2e\Sigma_{\lambda\nu}(x)-\left(\frac{1}{e}\int d^4z\Sigma_{\alpha\beta}(x)D^{\alpha\beta,\lambda\nu}(z-x)\right)\right]\right. \\ &\quad\quad\left.\times\left[2e\Sigma_{\mu\rho}(y)-\frac{1}{e}\int d^4u\Sigma_{\gamma\zeta}(u)D^{\gamma\zeta,\mu\rho}(|u-y|)\right]\right\rangle_\Sigma, \end{aligned} \tag{26}$$

where the normalized Gaussian path-integral average $\langle\,\rangle_\Sigma$ is defined explicitly by

$$\begin{aligned} \langle\,\rangle_\Sigma &= \frac{1}{2}\int D^F[\Sigma_{\alpha\beta}(x)](\dots)\exp\left[-\pi^2\int_{R^4}dx\,dy\,\Sigma_{\alpha\beta}(x)\right. \\ &\quad\left.\times D^{\alpha\beta,\gamma\zeta}(|x-y|)\Sigma_{\gamma\zeta}(y)\right] \end{aligned} \tag{27a}$$

with

$$Z=\int D^F[\Sigma_{\alpha\beta}(x)]\exp\left[-\pi^2\int_{R^4}dx\,dy\,\Sigma_{\alpha\beta}(x)D^{\alpha\beta,\gamma\zeta}(|x-y|)\Sigma_{\gamma\zeta}(y)\right]. \tag{27b}$$

After our remarks as expressed by (26)–(27) on the Antonov-Ebert paper [9], we now pass on to the Wilson Loop evaluation of (1a) on the dual "stringy effective" path integral (26)–(27).

The main point for evaluation of (1a) in terms of an effective string theory is to re-write it in terms of the strenght field by means of the Stokes Theorem followed obviously by the string path integral average (27).

We have thus the following string functional integral representation for the Wilson Loop

$$\langle W[C_{(R,T)}]\rangle = \left\langle \exp\left\{ie\int d^4x(F_{\alpha\beta}(A)\cdot J_{\alpha\beta}(C_{R,T})\right\}\right\rangle_{\Sigma}. \tag{28}$$

Here the boundary's rectangle Loop current

$$J_{\alpha\beta}(C_{R,T})(x^{\mu}) = \int_{C_{(R,T)}} \delta(x^{\mu} - X^{\mu}(s))(X_{\alpha}dX_{\beta})(s),$$

with $X_{\mu}(s)$ for $0 \le s \le 1$ denoting a parametrization of the rectangle's boundary $C_{(R,T)}$. At this point of our study, we propose to use a cumulant expansion for evaluating (28) in the "stringy" DAHM model, which in generic form reads

$$\begin{aligned}
&\langle W[C_{R,T}]\rangle \\
&= \left\langle \exp\left\{ie\int F\cdot J\right\}\right\rangle \\
&= \exp\left\{\left\langle ie\int F\cdot J\right\rangle_{\Sigma} + \frac{1}{2}\left[\left\langle\left(ie\int F\cdot J\right)^2\right\rangle_{\Sigma} - \left\langle\left(ie\int F\cdot J\right)^2\right\rangle_{\Sigma}\right] + \dots\right\}.
\end{aligned} \tag{29}$$

Extensive calculations of (29), including spin degrees of freedom (see [13, 14] and [22]) will be reported elsewhere.

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